Stat 155 Lecture 22 Notes

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April 24, 2018

1 Voting

1.1 Voting preferences, preference communication, and Borda count

What is a voting mechanism? What properties would we like voting mechanisms to have? What mechanisms possess these properties?

How do we model voters' preferences? How do voters express their preferences? How do we combine that information? We will distinguish two outcomes:

1. A single winner ("voting rule")

2. A ranking of all candidates ("ranking rule")

Here are some models for voters' preferences.

Example 1.1. Each voter has a ranking of the set of candidates: Voter i has a permutation π on Γ. However, this is not perfect. Sometimes an individual does not have a total order on the candidates. And sometimes and individual's preferences are not transitive (they might prefer A over B over C over A).

Example 1.2. Each voter has a utility associated with each candidate: Voter i has utility function $u_i : \Gamma \to \mathbb{R}$. This is a more fine-grained model: it allows us to compare the total utility of different outcomes. However, it is more difficult to assign scores than to compare, and scores are typically incomparable between individuals.

Voters can communicate their preferences in many ways.

Example 1.3. Here are some ways voters can communicate their preferences.

1. Each voter assigns a score to each candidate.

2. Each voter assigns a ranking of the set of candidates.

3. Each voter approves of a subset of the set of candidates.

4. Each voter approves of a single candidate.

Last lecture, we discussed some examples of voting systems. Here is another.

Example 1.4. Here is a voting system called the *Borda count.* Voters provide a ranking of the candidates, from 1 to $|\Gamma|$. A candidate that is ranked in the i-th position is assigned $|\Gamma| - i + 1$ points. Candidates are ranked by the total number of points assigned.

1.2 Properties of voting systems

What formal assumptions do we make when modeling voting? There is a set γ of candidates. Voter i has a preference relation $\succ_i$ defined on candidates that is:

1. Complete: for every $A \neq B$, $A \succ_i B$ or $B \succ_i A$.
2. Transitive: for every A, B, C, if $A \succ_i B$ and $B \succ_i C$, then $A \succ_i C$.

Definition 1.1. A *voting rule* is a function f that maps a preference profile $\pi = (\succ_1, \ldots, \succ_n)$ to a winner from Γ.

Definition 1.2. A *ranking rule* is a function R that maps a preference profile $\pi = (\succ_1, \ldots, \succ_n)$ to a social ranking $\triangleright$ on Γ, which is another complete, transitive preference relation.

While more than one candidate remains: Eliminate the bottom-ranked k candidates, Apply ranking rule to voters' preferences over remaining candidates.

Example 1.5. If $k = |\Gamma| - 1$, take the top-ranked candidate as the winner. If the ranking is based on voters' top choices this is plurality voting.

Example 1.6. If $k = |\Gamma| - 2$ and the ranking is based on the voters' top choices, this is contingent voting.

Example 1.7. If $k = 1$ and the ranking is based on voters' top choices, this is instant-runoff voting.

Definition 1.3. A ranking rule R has the *unanimity* property if, for all i, $A \succ_i B$, then $\triangleright = R(\succ_1, \ldots, \succ_n)$ satisfies $A \triangleright B$; i.e. if all voters prefer candidate A over B, then candidate A should be ranked above B.

It is hard to imagine a "fair" voting rule that violates unanimity.

Definition 1.4. A ranking rule R is *strategically vulnerable* if, for some preference profile $(\succ_1, \ldots, \succ_n)$, some voter i, some candidates A and B, and

$$\triangleright = R(\succ_1, \ldots, \succ_i, \ldots, \succ_n),$$

$$\triangleright' = R(\succ_1, \ldots, \succ_i', \ldots, \succ_n),$$

then $A \succ_i B$, and $B \triangleright A$, but $A \triangleright' B$.

This meas that Voter i has a preference relation $\succ_i$, but by stating an alternative preference relation $\succ_i'$, they can swap the ranking rule's preference between A and B to make it consistent with $\succ_i'$.

Definition 1.5. *Independence of irrelevant alternatives (IIA)* is the following property of a ranking rule R. If you have two different voter preference profiles $(\succ_1, \dots, \succ_n)$ and $(\succ_1', \dots, \succ_n')$, define $\triangleright := R(\succ_1, \dots, \succ_n)$ and $\triangleright' := R(\succ_1', \dots, \succ_n')R$. If, for all i, $A \succ_i B \iff A \succ_i' B$, then $A \triangleright B \iff A \triangleright' B$.

This says that the ranking rule's relative rankings of candidates A and B should depend only on the voters' relative rankings of these two candidates.

Example 1.8. Ranking based on runoff voting violates IIA. Consider the following example of strategic voting.

	1st	2nd	3rd
30%	A	B	C
45%	B	C	A
25%	C	A	B

Instant runoff gives us the ranking $A \triangleright B \triangleright C$. But if 10% of the people in the second group lie about their preferences, we get a different result.

	1st	2nd	3rd
30%	A	B	C
35%	B	C	A
10%	C	B	A
25%	C	A	B

Here, instant runoff gives us the ranking $B \triangleright C \triangleright A$. But when the 10% changed their preferences, they did not change their relative preferences between B and A. So IIA is violated.

1.3 Violating IIA and Arrow's Impossibility theorem

Theorem 1.1. *Any ranking rule R that violates IIA is strategically vulnerable.*

Proof. Suppose R violates IIA. Let $\pi = (\succ_1, \dots, \succ_n)$, $\pi' = (\succ_1', \dots, \succ_n')$, $\triangleright = R(\pi)$, and $\triangleright' = R(\pi')$. Then for all i, $A \succ_i B \iff A \succ_i' B$, but $A \triangleright B$ and $B \triangleright' A$. Change the voters' rankings one by one to change π into π':

$$\begin{array}{cc}
(\succ_1, \succ_2, \dots, \succ_n) & A \triangleright B \\
(\succ_1', \succ_2, \dots, \succ_n) & A \triangleright_1 B \\
(\succ_1', \succ_2', \dots, \succ_n) & B \triangleright_2 A \\
\vdots & \vdots \\
(\succ_1', \succ_2', \dots, \succ_n') & B \triangleright' A
\end{array}$$

Then some voter on the path from π to π' changes the order of A and B. So R is strategically vulnerable. $\square$

Definition 1.6. A ranking rule R is a dictatorship if there is a voter i^* such that, for any preference profile $(\succ_1, \dots, \succ_n)$ and $\triangleright = R(\succ_1, \dots, \succ_n)$, $A \triangleright B \iff A \succ_{i^*} B$.

Theorem 1.2 (Arrow's Impossibility theorem)**.** *For* $|\Gamma| \geq 3$*, any ranking rule* R *that satisfies both IIA and unanimity is a dictatorship.*

Corollary 1.1. *Any ranking rule* R *that satisfies unanimity and is not strategically vulnerable is a dictatorship.*

Violating unanimity does not make sense, but a dictatorship is undesirable.[1] Hence, strategic vulnerability is inevitable.

[1] You may disagree.